phull: p-hull in R

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Outline

Preliminaries

Given an arbitrary $0 < p < \infty$, $x_0, y_0 \in \mathbb{R}$, $a \geq 0$ and $b \geq 0$, let

$$E_{p,a,b}^{(x_0,y_0)} = \left\{ (x,y) \in \mathbb{R}^2 : \left| \frac{y - y_0}{b} \right|^p + \left| \frac{x - x_0}{a} \right|^p \leq 1 \right\}. \qquad (1)$$

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Moreover, for $p = \infty$ we have

$$E_{p,a,b}^{(x_0,y_0)} = \left\{ (x,y) \in \mathbb{R}^2 : \max \left\{ \left| \frac{y - y_0}{b} \right|, \left| \frac{x - x_0}{a} \right| \right\} \leq 1 \right\}, \quad (2)$$

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and for $p = 0$

$$E_{p,a,b}^{(x_0,y_0)} = \left\{ (x, y) \in \mathbb{R}^2 : \begin{array}{ll} & x \in [x_0 - a, x_0 + a] \quad \wedge \quad y = y_0 \\ \vee & y \in [y_0 - b, y_0 + b] \quad \wedge \quad x = x_0 \end{array} \right\}. \quad (3)$$

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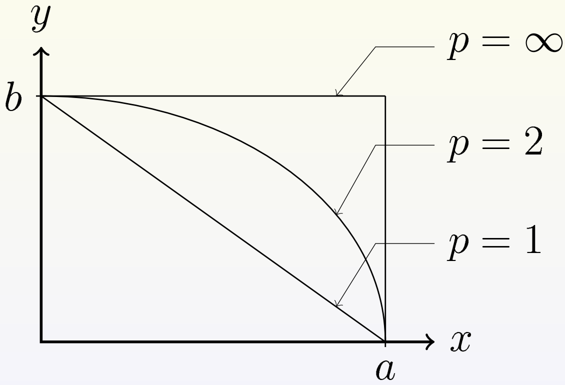
and for $p = 0$

$$E_{p,a,b}^{(x_0,y_0)} = \left\{(x,y) \in \mathbb{R}^2 : \begin{array}{ccc} & x \in [x_0 - a, x_0 + a] & \wedge \quad y = y_0 \\ \vee & y \in [y_0 - b, y_0 + b] & \wedge \quad x = x_0 \end{array}\right\}. \tag{3}$$

We call $E_{p,a,b}^{(x_0,y_0)}$ the p-**ellipse of size** (a,b) **centered at** (x_0, y_0).

Preliminaries

Illustration: $\partial E_{p,a,b}^{(0,0)} \cap \mathbb{R}_0^+ \times \mathbb{R}_0^+$.



Preliminaries

We are given a finite planar set $Q = \{q_1, q_2, \ldots, q_n\}$, such that $q_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 1, \ldots, n$ $(n \geq 4)$.

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Let

$$\begin{aligned} x_{\mathrm{l}} &= \min_{p_i \in P} x_i, \\ x_{\mathrm{r}} &= \max_{p_i \in P} x_i, \\ y_{\mathrm{b}} &= \min_{p_i \in P} y_i, \\ y_{\mathrm{t}} &= \max_{p_i \in P} y_i. \end{aligned}$$

Then $B(Q) = [x_{\mathrm{l}}, x_{\mathrm{r}}] \times [y_{\mathrm{t}}, y_{\mathrm{b}}]$ is the **minimal bounding rectangle** of Q.

Preliminaries

For a fixed $p \geq 0$ let

$$\begin{aligned}
C_p^{\mathrm{bl}}(Q) &= \bigcup\nolimits_{a,b\colon\ Q \notin \operatorname{int} E_{p,a,b}^{(x_\mathrm{l},y_\mathrm{b})}} E_{p,a,b}^{(x_\mathrm{l},y_\mathrm{b})}, \\
C_p^{\mathrm{br}}(Q) &= \bigcup\nolimits_{a,b\colon\ Q \notin \operatorname{int} E_{p,a,b}^{(x_\mathrm{r},y_\mathrm{b})}} E_{p,a,b}^{(x_\mathrm{r},y_\mathrm{b})}, \\
C_p^{\mathrm{tr}}(Q) &= \bigcup\nolimits_{a,b\colon\ Q \notin \operatorname{int} E_{p,a,b}^{(x_\mathrm{r},y_\mathrm{t})}} E_{p,a,b}^{(x_\mathrm{r},y_\mathrm{t})}, \\
C_p^{\mathrm{tl}}(Q) &= \bigcup\nolimits_{a,b\colon\ Q \notin \operatorname{int} E_{p,a,b}^{(x_\mathrm{l},y_\mathrm{t})}} E_{p,a,b}^{(x_\mathrm{l},y_\mathrm{t})}.
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\end{aligned}$$

We further on assume $\mathrm{int}\, C_p^{\mathrm{bl}}(Q)$, $\mathrm{int}\, C_p^{\mathrm{br}}(Q)$, $\mathrm{int}\, C_p^{\mathrm{tr}}(Q)$, $\mathrm{int}\, C_p^{\mathrm{tl}}(Q)$ are mutually exclusive.

p-hull

Definition

Let $Q = \{q_1, q_2, \ldots, q_n\} \subset \mathbb{R}^2$ and $p \geq 0$. The p-**hull** of Q, denoted by $H_p(Q)$, is defined by

$$H_p(Q) = \partial \left(B(Q) \setminus C_p^{\mathrm{bl}}(Q) \setminus C_p^{\mathrm{br}}(Q) \setminus C_p^{\mathrm{tr}}(Q) \setminus C_p^{\mathrm{tl}}(Q) \right). \tag{4}$$

Properties of a p-hull

Proposition

Let $Q = \{q_1, q_2, \ldots, q_n\} \subset \mathbb{R}^2$ *and* $p \geq 0$. *Then we have the following.*

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1. *If $p = 1$ then $H_p(Q)$ is the convex hull of Q.*

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2. *If* $p = \infty$ *then* $H_p(Q)$ *is the X-Y hull of* Q *(see Nicholl et al, 1983).*

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2. *If* $p = \infty$ *then* $H_p(Q)$ *is the X-Y hull of* Q *(see Nicholl et al, 1983).*
3. *If* $p = 0$ *then* $H_p(Q) = \partial B(Q)$.

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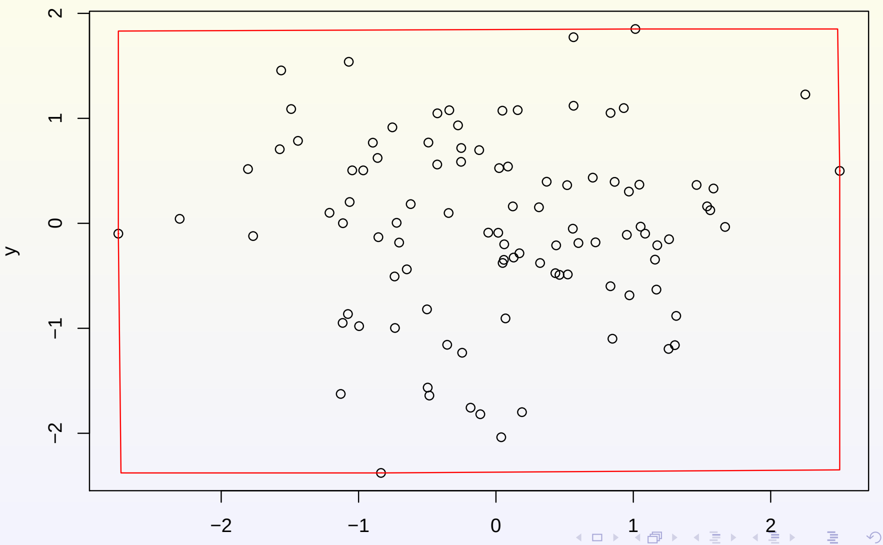
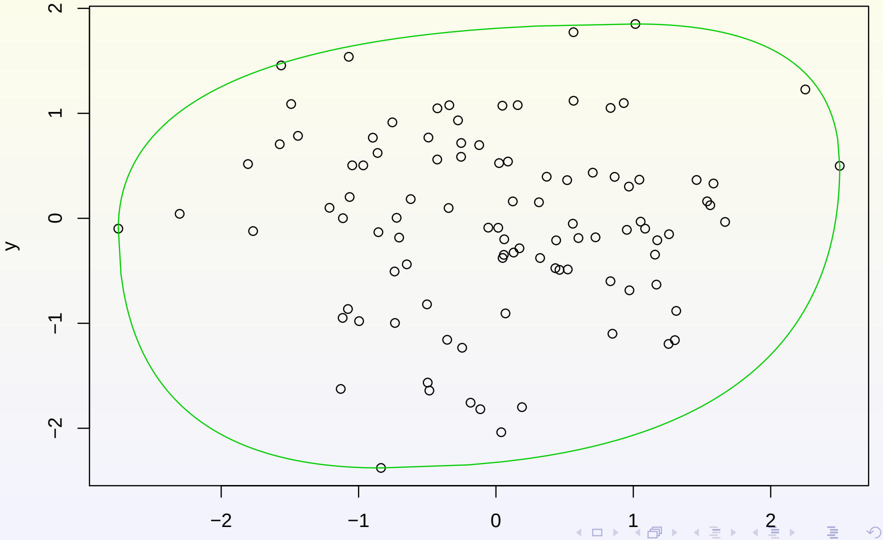
2. $H_p(Q)$ is not rotation-invariant (thus it is orientation-dependent) for $p \neq 1$.
3. $H_p(Q)$ is convex for $p \leq 1$.
4. If $p' \geq p$, then $H_{p'}(Q) \subseteq H_p(Q)$.

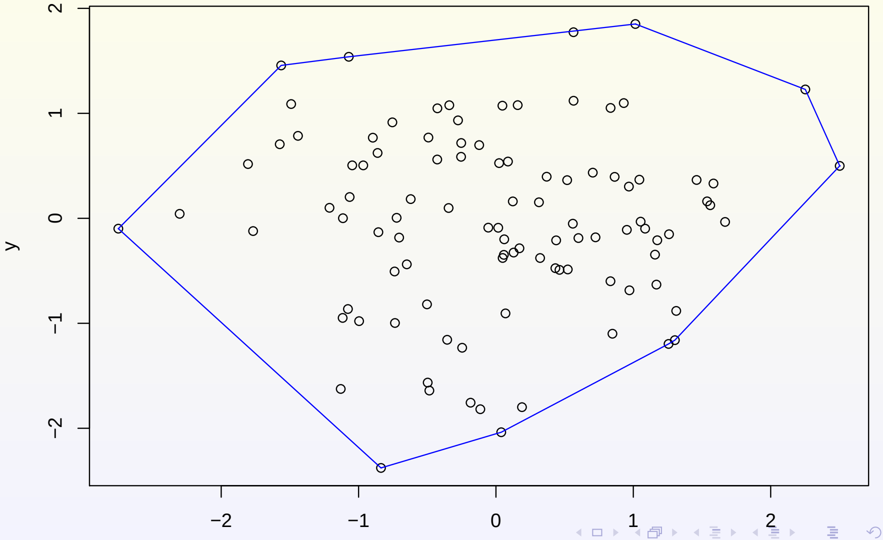
Example: $p = 0.1$



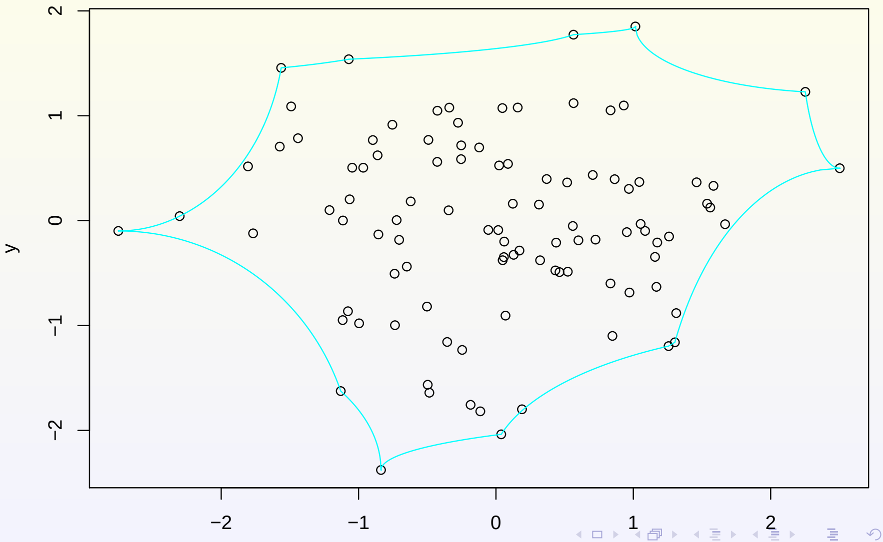
Example: $p = 0.5$



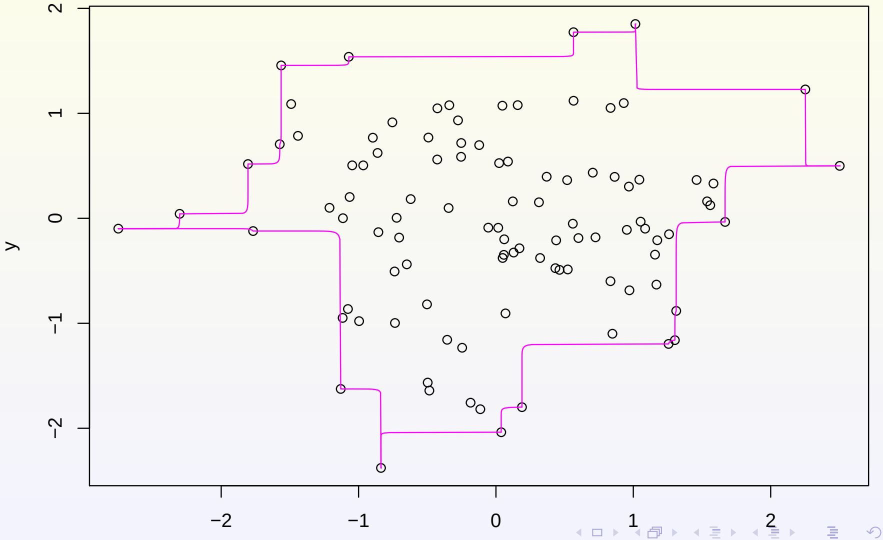
Example: $p = 1.0$



Example: $p = 2.0$



Example: $p = 50$



Computation

Let

$$
\begin{aligned}
q_{\mathrm{bl}_1} &= \operatorname*{arg\,min}_{q_i \in Q:\ x_i = x_\mathrm{l}} y_i, & q_{\mathrm{bl}_2} &= \operatorname*{arg\,min}_{q_i \in Q:\ y_i = y_\mathrm{b}} x_i, \\
q_{\mathrm{br}_1} &= \operatorname*{arg\,max}_{q_i \in Q:\ y_i = y_\mathrm{b}} x_i, & q_{\mathrm{br}_2} &= \operatorname*{arg\,min}_{q_i \in Q:\ x_i = x_\mathrm{r}} y_i, \\
q_{\mathrm{tr}_1} &= \operatorname*{arg\,max}_{q_i \in Q:\ x_i = x_\mathrm{r}} y_i, & q_{\mathrm{tr}_2} &= \operatorname*{arg\,max}_{q_i \in Q:\ y_i = y_\mathrm{t}} x_i, \\
q_{\mathrm{tl}_1} &= \operatorname*{arg\,min}_{q_i \in Q:\ y_i = y_\mathrm{t}} x_i, & q_{\mathrm{tl}_2} &= \operatorname*{arg\,max}_{q_i \in Q:\ x_i = x_\mathrm{l}} y_i.
\end{aligned}
$$

Note that all the points $\in \partial B(Q)$.

Computation (cont'd)

Decomposition:

$$H_p(Q) \;=\; \partial\left(B(Q) \setminus C_p^{\mathrm{bl}}(Q) \setminus C_p^{\mathrm{br}}(Q) \setminus C_p^{\mathrm{tr}}(Q) \setminus C_p^{\mathrm{tl}}(Q)\right)$$

Computation (cont'd)

Decomposition:

$$
\begin{aligned}
H_p(Q) &= \partial\left(B(Q) \setminus C_p^{\mathrm{bl}}(Q) \setminus C_p^{\mathrm{br}}(Q) \setminus C_p^{\mathrm{tr}}(Q) \setminus C_p^{\mathrm{tl}}(Q)\right) \\
&= \left(\partial C_p^{\mathrm{bl}}(Q) \cup \partial C_p^{\mathrm{br}}(Q) \cup \partial C_p^{\mathrm{tr}}(Q) \cup \partial C_p^{\mathrm{tl}}(Q)\right) \cap B(Q) \\
\cup\ & \overline{q_{\mathrm{bl}_2} q_{\mathrm{br}_1}} \cup \overline{q_{\mathrm{br}_2} q_{\mathrm{tr}_1}} \cup \overline{q_{\mathrm{tr}_2} q_{\mathrm{tl}_1}} \cup \overline{q_{\mathrm{tl}_2} q_{\mathrm{bl}_1}}.
\end{aligned} \tag{5}
$$

Computation (cont'd)

Decomposition:

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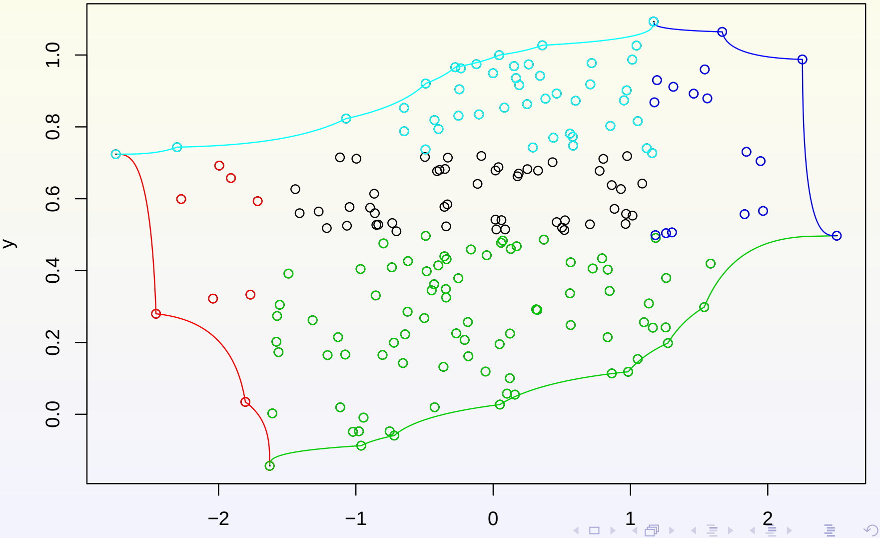
Moreover:

$$\begin{aligned} \partial C_p^{\mathrm{bl}}(Q) &= \partial C_p^{\mathrm{bl}}(\{q_i \in Q : \ x_i \le x_{\mathrm{bl}_2} \wedge y_i \le y_{\mathrm{bl}_1}\}), \\ \partial C_p^{\mathrm{br}}(Q) &= \partial C_p^{\mathrm{br}}(\{q_i \in Q : \ x_i \ge x_{\mathrm{br}_1} \wedge y_i \le y_{\mathrm{br}_2}\}), \\ \partial C_p^{\mathrm{tr}}(Q) &= \partial C_p^{\mathrm{tr}}(\{q_i \in Q : \ x_i \ge x_{\mathrm{tr}_2} \wedge y_i \ge y_{\mathrm{tr}_1}\}), \\ \partial C_p^{\mathrm{tl}}(Q) &= \partial C_p^{\mathrm{tl}}(\{q_i \in Q : \ x_i \le x_{\mathrm{tl}_1} \wedge y_i \ge y_{\mathrm{tl}_2}\}). \end{aligned} \tag{6}$$

Computation (cont'd)



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3. Output: $P \cap \partial C_p^{\mathrm{bl}}(Q)$ (without loss of generality).
4. Input: $p \geq 0$, $W = \{q_i \in Q : \ x_i \leq x_{\mathrm{bl}_2} \wedge y_i \leq y_{\mathrm{bl}_1}\}$ as an array **sorted** by x coordinate $w[1], \ldots, w[m]$.

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5. Denotation: by $E_{p,q_i,q_j}^{(x_0,y_0)}$ we mean an p-ellipse centered at (x_0, y_0) interpolating $q_i \neq q_j$.

Computation (cont'd)

Create an empty stack $\mathbf{S}$;
Push $w[1]$ into $\mathbf{S}$;
$i := 2$;
while $(i < n)$ **and** $(w[i]_y \geq w[1]_y)$ **do**
    $i := i + 1$;
Push $w[i]$ into $\mathbf{S}$;
for $j = i + 1, i + 2, \ldots, n$ **do**
    if $(\mathbf{S}[\#\mathbf{S}]_y < w[j]_y)$ **then** {
        while $(\#\mathbf{S} \geq 2)$ **and** $(\mathbf{S}[\#\mathbf{S} - 1] \in E_{p,\mathbf{S}[\#\mathbf{S}],w[j]}^{(x_{\mathrm{bl}},y_{\mathrm{bl}})})$ **do**
            Pop from $\mathbf{S}$;
        Push $w[j]$ into $\mathbf{S}$;
    }
return $\mathbf{S}$;

Computation (cont'd)

Implementation: `phull` 0.1-2 — package available on CRAN.
(http://cran.r-project.org/web/packages/phull/index.html)

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Example: axes rotation.

```
library(phull); # load the library
```

Computation (cont'd)

Implementation: `phull` 0.1-2 — package available on CRAN.
(http://cran.r-project.org/web/packages/phull/index.html)

Example: axes rotation.

```
library(phull); # load the library

translateAndRotate <- function(data, x0, y0, angle)
{ ... }

rotateAndTranslate <- function(data, x0, y0, angle)
{ ... }
```

Computation (cont'd)

```
set.seed(98765); n <- 1000; p <- 3.0;
data <- matrix(c(rnorm(n), rt(n, 10)), ncol=2); # input data
nres <- 50; # "resolution"
```

Computation (cont'd)

```
set.seed(98765); n <- 1000; p <- 3.0;
data <- matrix(c(rnorm(n), rt(n, 10)), ncol=2); # input data
nres <- 50; # "resolution"

ptest <- phull(data, p=p);                    # compute the p-hull
discr_0 <- as.matrix(ptest, nres=nres);       # sample
```

Computation (cont'd)

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set.seed(98765); n <- 1000; p <- 3.0;
data <- matrix(c(rnorm(n), rt(n, 10)), ncol=2); # input data
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ptest <- phull(data, p=p);                  # compute the p-hull
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print(ptest)
```

```
        p-hull, p=3

data: data
1000 points, bounding rectangle: (...)
```

Computation (cont'd)

```
data2 <- translateAndRotate(data, angle=-pi/6
   -ptest$xrange[1], -ptest$yrange[1]);
ptest2 <- phull(data2, p=p);                     # compute the p-hull
discr_30 <- as.matrix(ptest2, nres=nres);  # sample
discr_30 <- rotateAndTranslate(discr_30, angle=pi/6,
   ptest$xrange[1], ptest$yrange[1]);
```

Computation (cont'd)

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data2 <- translateAndRotate(data, angle=-pi/6
   -ptest$xrange[1], -ptest$yrange[1]);
ptest2 <- phull(data2, p=p);                # compute the p-hull
discr_30 <- as.matrix(ptest2, nres=nres);  # sample
discr_30 <- rotateAndTranslate(discr_30, angle=pi/6,
   ptest$xrange[1], ptest$yrange[1]);

plot(data, type="p", pch=1);
lines(discr_0,  col=2);
lines(discr_30, col=4);
```

Computation (cont'd)

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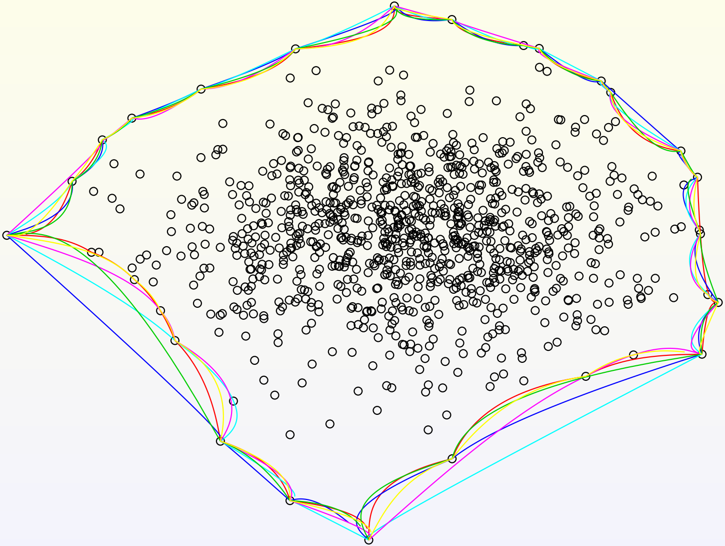
data2 <- translateAndRotate(data, angle=-pi/6
   -ptest$xrange[1], -ptest$yrange[1]);
ptest2 <- phull(data2, p=p);                  # compute the p-hull
discr_30 <- as.matrix(ptest2, nres=nres);  # sample
discr_30 <- rotateAndTranslate(discr_30, angle=pi/6,
   ptest$xrange[1], ptest$yrange[1]);

plot(data, type="p", pch=1);
lines(discr_0,  col=2);
lines(discr_30, col=4);
```

. . . and so on. . .

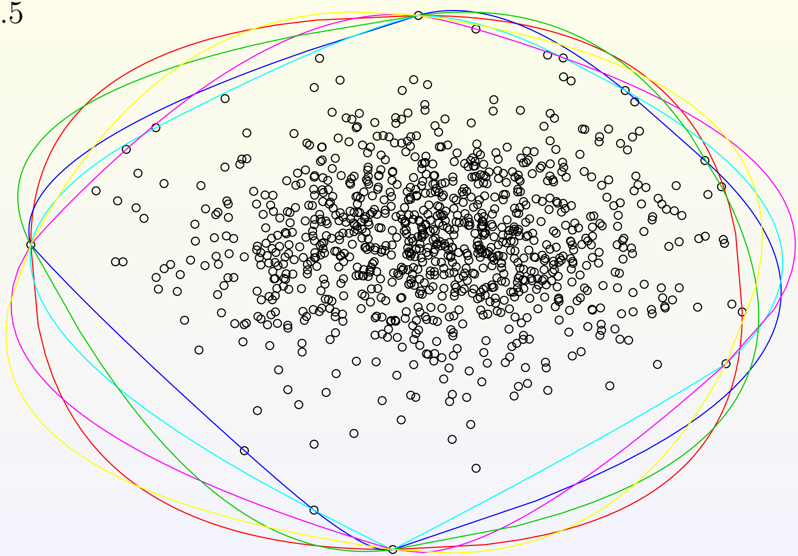
Computation (cont'd)

$p = 3$



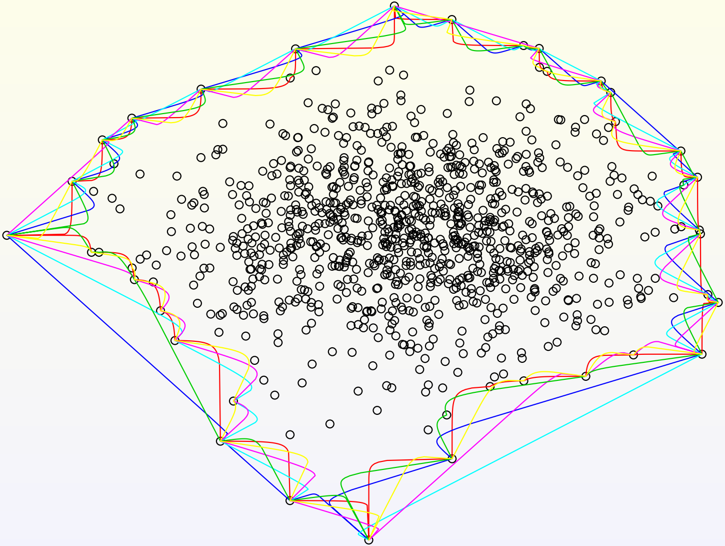
Computation (cont'd)

$p = 0.5$



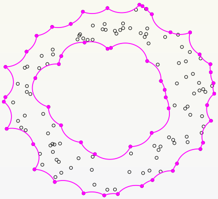
Computation (cont'd)

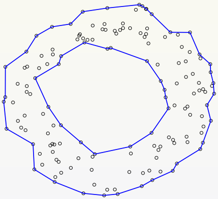
$p = 20$

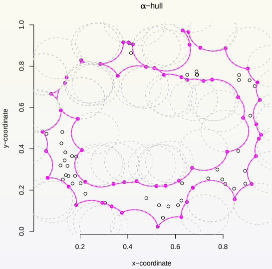


Related packages

`alphahull` (Pateiro-Lopez, Rodriguez-Casal, 2009): α-shapes (Edelsbrunner et al, 1983).







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Thank you for your attention.